ANSWERS

1. D
2. B
3. E (π/4)
4. B
5. C
6. B
7. D
8. E (76)
9. B
10. A
11. E (16/15)
12. A
13. A
14. A
15. A
16. B
17. D
18. A
19. D
20. A
21. D
22. D
23. A
24. C
25. B
26. E (2π)
27. A
28. B
29. B
30. B

SOLUTIONS

1. The graph of the function over the interval makes two right triangles. The corner of the function occurs at $x = \frac{1}{2}$, so one right triangle has area $\frac{1}{2}\left(\frac{1}{2} - (-1)\right)|2(-1) - 1| = \frac{9}{4}$ and the other has area $\frac{1}{2}\left(3 - \frac{1}{2}\right)|2 \cdot 3 - 1| = \frac{25}{4}$. Thus the integral is $\frac{9}{4} + \frac{25}{4} = \frac{34}{4} = \frac{17}{2} = 8.5$.

2. We have

$$\int_0^{\pi/4} \frac{\sin(x) - \cos(x)}{\cos(x)}\, dx = \int_0^{\pi/4} (\tan(x) - 1)\, dx = \ln|\sec(x)| - x\Big|_0^{\pi/4} = \ln\sqrt{2} - \frac{\pi}{4},$$

which can be rewritten as $\frac{1}{4}(\ln 4 - \pi)$.

3. By completing the square of the denominator, we have

$$\int_{-5}^{-1} \frac{1}{x^2 + 6x + 13}\, dx = \int_{-5}^{-1} \frac{1}{(x+3)^2 + 4}\, dx = \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)\Bigg|_{-5}^{-1},$$

which becomes $\frac{1}{2}\tan^{-1}(1) - \frac{1}{2}\tan^{-1}(-1) = \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi}{4}$.

4. Using integration by parts for the logarithmic term and the trigonometric identity $4\sin^2(x) = 2 - 2\cos(2x)$ for the trigonometric term, we can obtain the antiderivative:

$$\int_1^{\pi/2} (\ln(x) + 4\sin^2(x))\, dx = \int_1^{\pi/2} (\ln(x) + 2 - 2\cos(2x))\, dx = x\ln(x) - x + 2x - \sin(2x)\Big|_1^{\pi/2}.$$

Evaluating the antiderivative, we get

$$\frac{\pi}{2}\ln\frac{\pi}{2} + \frac{\pi}{2} - (1 - \sin(2)) = \frac{\pi}{2}\left(\ln\frac{\pi}{2} + 1\right) + \sin(2) - 1.$$

5. We use the shell method. The resulting antiderivative is found using integration by parts.

$$2\pi\int_1^e 3x^2\ln(x)\, dx = 2\pi\left(x^3\ln(x)\Big|_1^e - \int_1^e x^2\, dx\right) = 2\pi\left(x^3\ln(x) - \frac{x^3}{3}\right)\Bigg|_1^e$$

This evaluates to

$$2\pi\left(e^3 - \frac{e^3}{3} - \frac{1}{3}\right) = \frac{2\pi}{3}(2e^3 + 1).$$

6. Displacement is simply the definite integral of the velocity between the indicated times. Thus, the displacement is

$$\int_{3^{-1/2}}^1 v(t)\, dt = 3t^4 - 4t^2 - t^{-4}\Big|_{3^{-1/2}}^1 = 3 - 4 - 1 - \left(\frac{1}{3} - \frac{4}{3} - 9\right) = -2 - (-10) = 8.$$

7. The length of a polar curve $r = r(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by the integral

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta.$$

Since $r = e^{\theta}$, then $\frac{dr}{d\theta} = e^{\theta}$ as well. Then our integral becomes

$$\int_0^{2\pi} \sqrt{e^{2\theta} + e^{2\theta}}\ d\theta = \sqrt{2}\int_0^{2\pi} e^{\theta}\ d\theta = \sqrt{2}(e^{2\pi} - 1).$$

The length is clearly greater than $e^{2\pi}$. Since $e^{5\pi/2} = e^{2\pi}e^{\pi/2}$ and $e^{\pi/2} > \sqrt{2}$, the length is also less than $e^{5\pi/2}$.

8. Integrating once, we find that $\frac{dy}{dx} = 2x^2 - 5x + K$. Using the initial value of the derivative, we compute $K = 3$. Integrating once more, we have $y = \frac{2}{3}x^3 - \frac{5}{2}x^2 + 3x + C$. Using the initial value of the function, we compute $C = 4$. Finally, $y(6) = 144 - 90 + 18 + 4 = 76$.

9. We have

$$\int_2^4 \frac{f'(x)\,dx}{4 + (f(x))^2} = \frac{1}{2}\tan^{-1}\left(\frac{f(x)}{2}\right)\Bigg|_2^4 = \frac{1}{2}(\tan^{-1}(-1) - \tan^{-1}(1)) = \frac{1}{2}\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = -\frac{\pi}{4}.$$

10. We have

$$\int_2^4 \frac{f(x)f'(x)\,dx}{4 + (f(x))^2} = \frac{1}{2}\ln\left(4 + \left(f(x)\right)^2\right)\Big|_2^4 = \frac{1}{2}(\ln 8 - \ln 8) = 0.$$

11. We set the difference of the equations equal to zero to get the intersection points. This yields $x^4 - 2x^2 + 1 = 0$ which factors into $(x^2 - 1)^2 = 0$. The solutions are $x = -1$ and $x = 1$. Then we integrate the difference of the functions between the intersection points. We have

$$\int_{-1}^{1} (x^4 - 2x^2 + 1)\,dx = \frac{x^5}{5} - \frac{2x^3}{3} + x\Bigg|_{-1}^{1} = \frac{1}{5} - \frac{2}{3} + 1 + \frac{1}{5} - \frac{2}{3} + 1 = \frac{16}{15}.$$

12. We use integration by parts to get the antiderivative. This gives us

$$\int_1^3 xf''(x)\ dx = xf'(x) - \int_1^3 f'(x)\ dx = xf'(x) - f(x)|_1^3 = 3(-2) - 3 - 0 + 1 = -8.$$

13. This is a Riemann sum:

$$\lim_{k\to\infty} \sum_{n=1}^{k} \frac{n^{2017}}{k^{2018}} = \lim_{k\to\infty} \frac{1}{k}\sum_{n=1}^{k}\left(\frac{n}{k}\right)^{2017} = \lim_{k\to\infty}\frac{1}{k}\left(\left(\frac{1}{k}\right)^{2017} + \left(\frac{2}{k}\right)^{2017} + \left(\frac{3}{k}\right)^{2017} + \cdots + 1\right).$$

This Riemann sum represents the definite integral of the function $f(x) = x^{2017}$ over the interval $[0, 1]$. This value is 1/2018.

14. The derivative is, by the Fundamental Theorem of Calculus,

$$f'(x) = e^{-(24x^2-8x^3-6x^4)^2}(48x - 24x^2 - 24x^3).$$

Setting this zero to get critical points implies that the exponential term will not contribute any critical points. Hence, we solve

$$48x - 24x^2 - 24x^3 = -24x(x^2 + x - 2) = -24x(x+2)(x-1) = 0$$

to get critical points of $x = -2$, $x = 0$, and $x = 1$. Using the First Derivative Test reveals that, since $f'(x) > 0$ for $x < -2$, $f'(x) < 0$ for $-2 < x < 0$, $f'(x) > 0$ for $0 < x < 1$, and $f'(x) < 0$ for $x > 1$, we have local maxima at both $x = -2$ and $x = 1$. However, the function computes an area under a curve. The maximum of the function is therefore the largest area. This implies that we want the largest value of the upper limit of integration. So we compute the upper limit: When $x = -2$, we obtain $24(-2)^2 - 8(-2)^3 - 6(-2)^4 = 48 + 64 - 96 = 16$; when $x = 1$, we obtain $24 - 8 - 6 = 10$. Hence, the maximum value of the function is given by $x = -2$.

15. Splitting up the integral, we have

$$\int_5^9 \left(3x^2 + 4f(x)\right)\,dx = \int_5^9 3x^2\,dx + 4\int_5^9 f(x)\,dx = x^3\big|_5^9 + 4\left(\int_2^9 f(x)\,dx - \int_2^5 f(x)\,dx\right).$$

This evaluates to $9^3 - 5^3 + 4(-2-7) = 729 - 125 - 36 = 568$.

16. Since $5x^2 = 7$ at $x = \pm\sqrt{\frac{7}{5}}$, these are the limits of integration. However, we can make our calculations a bit easier by using a lower limit of 0 (since the region is symmetric about the y-axis) and doubling the integral. The area between the line and the parabola is

$$2\int_0^{\sqrt{7/5}} (7 - 5x^2)\,dx = 2\left(7x - \frac{5x^3}{3}\right)\Bigg|_0^{\sqrt{7/5}} = 2\left(7\sqrt{\frac{7}{5}} - \frac{5}{3}\cdot\frac{7}{5}\sqrt{\frac{7}{5}}\right) = \frac{28}{3}\sqrt{\frac{7}{5}}.$$

Now, we want the area between the line $y = a$ and the parabola to be half of the above area. In this case, the upper limit of integration is $\sqrt{\frac{a}{5}}$. So we solve

$$2\int_0^{\sqrt{a/5}} (a - 5x^2)\,dx = \frac{14}{3}\sqrt{\frac{7}{5}}.$$

The value of the definite integral in terms of a is

$$2\int_0^{\sqrt{a/5}} (a - 5x^2)\,dx = 2\left(ax - \frac{5x^3}{3}\right)\Bigg|_0^{\sqrt{a/5}} = 2\left(a\sqrt{\frac{a}{5}} - \frac{5}{3}\cdot\frac{a}{5}\sqrt{\frac{a}{5}}\right) = \frac{4a}{3}\sqrt{\frac{a}{5}}.$$

So now we solve $\frac{4a}{3}\sqrt{\frac{a}{5}} = \frac{14}{3}\sqrt{\frac{7}{5}}$ which simplifies to $4a^3 = 7^3$. Therefore, $a = \frac{7}{\sqrt[3]{4}} = \frac{7}{2}\sqrt[3]{2}$.

17. We have the following.

$$\int_0^2 \frac{\left(1 + g'(x)\right)\left(f'\left(x + g(x)\right)\right)}{\sqrt{f\left(x + g(x)\right)}}\,dx = 2\sqrt{f\left(x + g(x)\right)}\Bigg|_0^2 = 2\sqrt{f(2)} - 2\sqrt{f\left(g(0)\right)}$$

Using values in the table, this evaluates to $2\sqrt{9} - 2\sqrt{4} = 6 - 4 = 2$.

18. We use a substitution. Let $u = x - 1$. Then $x = u + 1$, $dx = du$, and the limits of integration become $-1 < u < 1$. The integral becomes

$$\int_0^2 x\sqrt{2x - x^2}\ dx = \int_0^2 x\sqrt{1 - (x-1)^2}\ dx = \int_{-1}^1 (u+1)\sqrt{1 - u^2}\ du.$$

We evaluate this by splitting up the integral:

$$\int_{-1}^1 u\sqrt{1-u^2}\ du + \int_{-1}^1 \sqrt{1-u^2}\ du.$$

The first integral has a straightforward antiderivative; the second is the area of semicircle of radius 1. Therefore, we get

$$\int_{-1}^1 u\sqrt{1-u^2}\ du + \int_{-1}^1 \sqrt{1-u^2}\ du = -\frac{1}{3}(1-u^2)^{3/2}\Big|_{-1}^1 + \frac{\pi}{2} = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

19. Note that $f^{-1}(0) = f(0)$ and that $f(1) = 4$ so that $f^{-1}(4) = 1$. The area between $f^{-1}(x)$ and the x-axis is equal to the area between $f(x)$ and the y-axis. We can easily compute the area between $f(x)$ and the y-axis by computing the area between $f(x)$ and the x-axis and subtracting it from the rectangle formed by $[0,1] \times [0,4]$, which has area 4. Thus,

$$\int_0^4 f^{-1}(x)\ dx = 4 - \int_0^1 f(x)\ dx = 4 - \left(\frac{x^6}{6} - \frac{x^4}{4} + 2x^2\right)\Big|_0^1 = 4 - \left(\frac{1}{6} - \frac{1}{4} + 2\right) = 4 - \frac{23}{12} = \frac{25}{12}.$$

20. We have $f'(x) = \cos(x)/(1 + \sin(x))$ and

$$f''(x) = \frac{-\sin(x)(1+\sin(x)) - \cos^2(x)}{(1+\sin(x))^2} = \frac{-\sin(x) - 1}{(1+\sin(x))^2} = -\frac{1}{1+\sin(x)}$$

so that $f(0) = 0$, $f'(0) = 1$ and $f''(0) = -1$. Then the first two nonzero terms of the Maclaurin series are $f(x) \approx x - x^2/2$. Integrating, we arrive at the approximation

$$\int_0^{1/4} \ln(1+\sin(x))\ dx \approx \int_0^{1/4}\left(x - \frac{x^2}{2}\right) dx = \frac{x^2}{2} - \frac{x^3}{6}\Big|_0^{1/4} = \frac{1}{32} - \frac{1}{384} = \frac{11}{384}.$$

21. Note that $1/(1 + x^{50})$ can be interpreted as the sum of a geometric series with first term 1 and common ratio $-x^{50}$. Thus, we can write

$$\frac{1}{1+x^{50}} = 1 - x^{50} + x^{100} - x^{150} + \cdots.$$

Using only the first two terms of the series as an approximation, we obtain

$$\int_0^1 \frac{1}{1+x^{50}}\ dx \approx \int_0^1 (1 - x^{50})\ dx = x - \frac{x^{51}}{51}\Big|_0^1 = 1 - \frac{1}{51} = \frac{50}{51}.$$

Since $0.98 = \frac{49}{50} < \frac{50}{51} < 1$, the exact value is between 0.97 and 1. (We are assured that we only need the first two terms since the series is alternating—by the Alternating Series Test, the error is not greater than the next unused term: $\frac{1}{101} < 0.01$.)

22. By splitting the fraction, we can integrate:

$$\int \frac{12x^5 - 8x^3 + 2}{x^3}\ dx = \int (12x^2 - 8 + 2x^{-3})\ dx = 4x^3 - 8x - x^{-2} + C = \frac{4x^5 - 8x^3 - 1 + Cx^2}{x^2}.$$

Because C can be any constant, an antiderivative is $\frac{4x^5 - 8x^3 - 1 + 5x^2}{x^2}$.

23. The numerator wants to factor, so let's help it! Add 1 and subtract 1 in the numerator. This allows us to write $x^4 + 4x^3 + 6x^2 + 4x + 1 - 1 = (x+1)^4 - 1$ as the numerator. Then

$$\int_0^1 \frac{(x+1)^4 - 1}{(x+1)^4}\ dx = \int_0^1 (1 - (x+1)^{-4})\ dx = x + \frac{1}{3(x+1)^3}\Big|_0^1 = 1 + \frac{1}{24} - \frac{1}{3} = \frac{17}{24}.$$

24. Note that $\int_1^2 f(x)\ dx$ is a constant; call it k. Then $f(x) = 16x^3 - 15x^2 + 2kx - 21$. However, we know that the definite integral of this function from 1 to 2 is equal to k. Thus, we have

$$k = \int_1^2 f(x)\ dx = 4x^4 - 5x^3 + kx^2 - 21x|_1^2 = 64 - 40 + 4k - 42 - 4 + 5 - k + 21 = 4 + 3k.$$

Solving $k = 4 + 3k$ yields $k = -2$. Thus $f(x) = 16x^3 - 15x^2 - 4x - 21$ so that $f(2) = 128 - 60 - 8 - 21 = 39$.

25. A little rewriting reveals that this limit is a Riemann sum:

$$\lim_{n\to\infty} \frac{1 + \sqrt[n]{e} + \sqrt[n]{e^2} + \cdots + \sqrt[n]{e^{n-1}}}{n} = \lim_{n\to\infty} \frac{1}{n}\left(1 + e^{1/n} + e^{2/n} + \cdots + e^{(n-1)/n}\right) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n} e^{k/n}.$$

This sum represents the definite integral of the function $F(x) = e^x$ over the interval $[0, 1]$. Hence its value, because the antiderivative of e^x is e^x, is simply $e - 1$.

26. Note that the integrand factors into $\sqrt{(x-3)(7-x)}$. We use the substitution $x = u + 5$ to then write the integrand as $\sqrt{(u+5-3)(7-u-5)} = \sqrt{(u+2)(2-u)} = \sqrt{4-u^2}$. This changes the limits of integration from $x = 3$ and $x = 7$ to $u = -2$ and $u = 2$. Hence, the integrand represents a semicircle of radius 2, and the integral's value is simply the area of the semicircle, which is 2π.

27. Multiply the numerator and denominator by $4 - x$ so that the integrand becomes

$$\sqrt{\frac{4-x}{4+x}} = \sqrt{\frac{(4-x)^2}{16-x^2}} = \frac{4-x}{\sqrt{16-x^2}} = \frac{4}{\sqrt{16-x^2}} - \frac{x}{\sqrt{16-x^2}}.$$

Now we can find and evaluate the antiderivative:

$$\int_0^4 \sqrt{\frac{4-x}{4+x}}\ dx = \int_0^4 \frac{4}{\sqrt{16-x^2}}\ dx - \int_0^4 \frac{x}{\sqrt{16-x^2}}\ dx = 4\sin^{-1}\left(\frac{x}{4}\right) - \sqrt{16-x^2}\Big|_0^4 = 4\cdot\frac{\pi}{2} - 4,$$

which simplifies to $2\pi - 4 = 2(\pi - 2)$.

28. We use the substitution $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\ d\theta$ and the limits of integration become $\theta = 0$ and $\theta = \pi/3$. Hence, we compute

$$\int_0^{\pi/3} \frac{\sqrt{4\sec^2\theta - 4}}{2\sec\theta} 2\sec\theta\tan\theta\ d\theta = \int_0^{\pi/3} 2\tan^2\theta\ d\theta = 2\int_0^{\pi/3} (\sec^2\theta - 1)\ d\theta.$$

The antiderivative evaluates to

$$2(\tan\theta - \theta)|_0^{\pi/3} = 2\left(\sqrt{3} - \frac{\pi}{3}\right) = 2\sqrt{3} - \frac{2\pi}{3}.$$

29. We first rewrite the integrand. This yields

$$\frac{-3x^2+6x-2}{3x^2-2x}=\frac{-(3x^2-2x)+4x-2}{3x^2-2x}=-1+\frac{4x-2}{3x^2-2x}$$

and then we use partial fractions to decompose the remaining fraction. We have

$$-1+\frac{4x-2}{3x^2-2x}=-1+\frac{1}{3x-2}+\frac{1}{x}.$$

Now we can determine the antiderivative

$$\int_{-2}^{-1}\frac{-3x^2+6x-2}{3x^2-2x}\,dx=\int_{-2}^{-1}\left(-1+\frac{1}{3x-2}+\frac{1}{x}\right)\,dx=-x+\frac{1}{3}\ln|3x-2|+\ln|x|\Big|_{-2}^{-1}$$

and evaluate the integral. We obtain

$$1+\frac{1}{3}\ln(5)-\left(2+\frac{1}{3}\ln(8)+\ln(2)\right)=\frac{1}{3}\ln\left(\frac{5}{8}\right)-\ln(2)-1=\ln\left(\frac{\sqrt[3]{5}}{2}\right)-\ln(2)-1=\ln\left(\frac{\sqrt[3]{5}}{4}\right)-1.$$

30. Using partial fractions and a trigonometric identity on the integrand yields the following.

$$\frac{1}{\sin(x)\,(1+\sin(x))}=\frac{1}{\sin(x)}-\frac{1}{1+\sin(x)}=\csc(x)-\frac{1-\sin(x)}{\cos^2(x)}$$

Using more trigonometric identities results in

$$\csc(x)-\frac{1-\sin(x)}{\cos^2(x)}=\csc(x)-\sec^2(x)+\frac{\sin(x)}{\cos^2(x)}=\csc(x)-\sec^2(x)+\sec(x)\tan(x).$$

Now we finally arrive at a point where we can find the antiderivative:

$$\int_{\pi/6}^{\pi/3}\left(\csc(x)-\sec^2(x)+\sec(x)\tan(x)\right)\,dx=-\ln|\csc(x)+\cot(x)|-\tan(x)+\sec(x)|_{\pi/6}^{\pi/3}.$$

Evaluation yields

$$-\ln\left|\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right|-\sqrt{3}+2+\ln\left|2+\sqrt{3}\right|+\frac{\sqrt{3}}{3}-\frac{2\sqrt{3}}{3}=\ln\left(\frac{2}{3}\sqrt{3}+1\right)-\frac{4\sqrt{3}}{3}+2.$$